Exact solution of a Riemann problem for pressure system arising in splitting numerical schemes for blood flow models

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Abstract

1D blood flow models have been extensively used to study wave propagation phenomena in arteries. More recently, their use has been extended to the highly deformable, i.e. highly non-linear, veins. In this work we build upon the flux vector splitting approach of Toro and Vázquez, hereafter called the TV splitting [1], originally developed for the conservative Euler equations of compressible gas dynamics. In this approach the flux vector is split into pressure and advection terms: in this manner one ends up with two systems of partial differential equations, one pressure system and one advection system. A change is made at PDEs level and regards the flux of the continuity equation: in the TV splitting approach this flux is assigned to the advection system, here it is assigned to the pressure system; this choice could be justified by the fact that in the construction of zero-dimensional simplifications to the equations in which the inertial terms are neglected, the continuity equation is preserved in the present formulation [2]. An advection equation for the concentration of a passive scalar is added, at PDEs level its conservative flux is assigned to the advection system for simplicity.

From the mathematical as well as numerical points of view, a basic problem to be solved is the special Cauchy problem called the Riemann problem [3]. This is of paramount importance because it provides an analytical solution to evaluate the numerical methods under study. In this work we present the exact solution of the Riemann problem for the pressure system, with the mathematical analysis and a numerical procedure to compute it. In particular the pressure system for veins results in the loss of genuine-non-linearity that leads to the formation of rarefactions, shocks and compound waves [4], these latter being a composition of rarefactions and shocks. We describe the different wave patterns and propose an algorithm for the exact solution of the Riemann problem for different wave configurations. We close the presentation with a series of numerical tests in case of arteries and veins, in which we compare the exact solution presented in this work with results obtained with classical centered and upwind numerical schemes.

References

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