Title: A-priori error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar fractional conservation laws

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Abstract: We consider the scalar fractional conservation law (FCL)

$$\partial_t u + \partial_x f(u) = -(-\partial_x^2)^{\lambda/2}(u) \qquad \text{in} \quad \mathbb{R} \times (0,T), \tag{1}$$

where $-(-\partial_x^2)^{\lambda/2}$ is the nonlocal fractional Laplace operator with $\lambda \in (0,1)$. FCLs are generalizations of convection-diffusion equations where the local diffusion may depend on the global dynamics. The fractional Burgers' equation, for example, is utilized when modeling detonation of gases that are driven by anomalous diffusion behaviour which can be described by means of the fractional Laplacian.

It is well known that solutions to FCLs may develop shocks in finite time if the diffusion fails to counterbalance the convection. Accordingly, in the context of numerical methods, low convergence rates have to be expected and have indeed already been proven by Cifani et al. [1]. Contrary to these worst-case estimates, one may be interested in achievable convergence rates when dealing with sufficiently smooth solutions to obtain a more differentiated picture of the numerical performance. This line of reasoning has also been followed in the setting of pure conservation laws $\partial_t u + \partial_x f(u) = 0$, which share some properties with FCLs.

In fact, for smooth solutions to conservation laws, Zhang and Shu [2] derived the a-priori error estimate

$$\|u(t^n) - u_h^n\|_{L^2(\mathbb{R})} \lesssim h^{k+1} + \tau^2, \tag{2}$$

for a second order Runge-Kutta discontinuous Galerkin (RKDG) method by using Taylor expansion and energy estimates, ultimately relying on the Gauss-Radau projection. Note that here and in the following, h is the maximal element width, τ is the time-step, and the numerical solution u_h^n at time t^n is sought in the space $\mathbb{P}^k$ of broken polynomials of degree k

At each time level t^n, Zhang and Shu introduce a projection operator $\mathbb{R}_h$ that actually depends on the exact solution $u^n = u(t^n)$: If $f'(u^n)$ is positive on any given element, interpolation on Radau points including the right boundary point is used on that element. Conversely, if $f'(u^n)$ is negative, interpolation on Radau points including the left boundary point is used. Finally, if $f'(u^n)$ changes sign on some element then the standard L^2 projection is used on that element.

The study of characteristics shows that if $f'(u(x,0)) = 0$ for some $x \in \mathbb{R}$, then $f'(u(x,t)) = 0$ for all $t > 0$ such that on each interval where $f'(u^0)$ is positive (negative/changes sign), the same is true for $f'(u^n)$. In particular, the projection operator $\mathbb{R}_h$ of Zhang and Shu does not depend on time and one has the important approximation property

$$\|\eta_u^{n+1} - \eta_u^n\|_{L^2(\mathbb{R})} \lesssim h^{k+1}\tau, \tag{3}$$

for the projection error $\eta_u^n := \mathbb{R}_h(u^n) - u^n$. In a subsequent Gronwall Lemma, this leads to an error of magnitude h^{k+1}.

In the context of FCLs, the characteristics argument is not viable and thus one must consider a time dependent projection operator $\mathbb{R}_h^n$, in general. As a consequence, we are left with only

$$\|\eta_u^{n+1} - \eta_u^n\|_{L^2(\mathbb{R})} \lesssim h^{k+1}, \tag{4}$$

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for times t^n at which $\mathbb{R}_h^{n+1} \neq \mathbb{R}_h^n$, in contrast to (3). Due to summation over all time-steps in the anticipated Gronwall Lemma, the approximation property (4) would yield an error of magnitude $h^{k+1}\tau^{-1}$.

In our work, we modify the projection operator, still denoted by $\mathbb{R}_h^n$, to accommodate a tolerance w.r.t. the mesh width h. By the regularity assumptions on the flux f and the exact solution u we are then able to effectively gauge the number of times N_j for which $\mathbb{R}_h^{n+1} \neq \mathbb{R}_h^n$ on any element I_j. This ultimately leaves us with improved bounds in the Gronwall argument and consequently in the a priori estimate:

Theorem. *Let $\alpha \geq 2$, $f \in C^{\alpha+1}(\mathbb{R})$ and $u \in C^{\alpha}([0,T]; C^1(\mathbb{R})) \cap C([0,T]; H^{k+1}(\mathbb{R}))$ be the exact solution to the FCL (1), then we have*

$$\|u^n - u_h^n\|_{L^2(\mathbb{R})} + \left(\sum_{m=0}^{n-1} \tau |u^m - u_h^m|^2_{H^{\lambda/2}(\mathbb{R})} \right)^{1/2} \lesssim h^{k+1}\tau^{-\frac{c(k)}{2\alpha}} + h^{k+1-\frac{\lambda}{2}} + \tau^2, \tag{5}$$

where u_h is the numerical solution given by our upwind RKDG scheme with second order TVD Runge-Kutta time discretization and $c(k)$ determines the CFL condition, i.e. we impose the time-step restriction $\tau^{c(k)} \lesssim h$ with $c(1) = 1$ and $c(k) = 3/4$ for all $k \geq 2$.

References:

[1] Simone Cifani, Espen R. Jakobsen and Kenneth H. Karlsen: The discontinuous Galerkin method for fractal conservation laws, 2011, IMA Journal of Numerical Analysis

[2] Qhiang Zhang and Chi-Wang Shu: Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws, 2006, ESAIM Mathematical Modelling and Numerical Analysis