On the connection between well-posedness theory and a posteriori error estimates for numerical methods for hyperbolic conservation laws

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In this talk, we will review some results on rigorous “reliable” a posteriori error estimates for numerical approximations of systems of hyperbolic conservation laws, i.e. bounds for discretization errors that can be computed from numerical solutions without making assumptions on the properties of the exact solution. We will explain the fundamental link between a posteriori error estimates and stability properties of the PDE that is to be approximated.

For hyperbolic consvervation laws, the literature on a posteriori error estimates is very limited and we will argue that this is closely connected with a lack of uniqueness and stability results for hyperbolic conservation laws, which in turn reflects the ill-posedness of systems in multiple space dimensions (for a large class of initial data).

We will begin by briefly describing a posteriori error estimates that have been derived for scalar problems based on L^1-contraction and Kruzkhov's doubling of variables technique and outline results for systems, obtained a few years ago, based on relative entropy stability estimates [2, 4]. The scope of the latter is limited in so far as they do not provide informative bounds in case the exact solution is discontinuous.

We will describe recent progress in a posteriori error estimates for systems of hyperbolic conservation laws in one spatial dimension. These results were obtained in joined work with Sam G. Krupa (Leipzig) and A. Sikstel (Cologne). They are based on two approaches: Firstly, results using Bressan's stability theory [1] and, secondly, results using a-contraction estimates based on work of Vasseur and Krupa [3].

The results based on [1] provide estimates in the $L^\infty(0, T, L^1(\mathbb{R}))$ norm and require to compute (local) residuals of the numerical solution in the $W^{-1,1}$ norm. We will show how this can be done exactly (for first order finite volume schemes) by applying a suitable projection to test functions.

The results based on [3] are more intricate in that they require significant modifications to the numerical schemes under consideration. At the same time they provide not only $L^\infty(0, T, L^1(\mathbb{R}))$ but also estimates on the position of shocks on error measured in $L^\infty(0, T : L^\infty(\Omega_s))$ where Ω_s is a set for which we know that it does not contain any shocks (but that is allowed to be in the wave cone of shocks).

References

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[2] A. Dedner, J. Giesselmann. A posteriori analysis of fully discrete method of lines discontinuous Galerkin schemes for systems of conservation laws. *SIAM J. Numer. Anal.* **54**, No. 6, 3523–3549, 2016.

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