Title: The mean field limit for 1D opinion dynamics with Coulomb interaction through front tracking approximations

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Abstract: We consider a model for opinion dynamics in form of a particle system on the line, previously studied in [1]. The system takes the form

$$\begin{cases} \dot{x}_i^N(t) = -\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} m_j^N(t) V'\big(x_i^N(t) - x_j^N(t)\big), & x_i^N(0) = \bar{x}_i^N, \\ \dot{m}_i^N(t) = -m_i^N(t)\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} m_j^N(t) S\big(x_i^N(t) - x_j^N(t)\big), & m_i^N(0) = \bar{m}_i^N \end{cases} \tag{1}$$

for $i \in \{1, \dots, N\}$. Here the scalar quantities x_i^N and m_i^N are respectively called the *opinions* and *weights*, while $V \colon \mathbb{R} \to \mathbb{R}$ is called an *interaction potential* and $S \colon \mathbb{R} \to \mathbb{R}$ is a function determining the evolution of weights. In order to preserve the total mass of the weights, which add up to N, we will assume S to be odd, and to further simplify the analysis we take it to be smooth and compactly supported, i.e., $S \in C_c^\infty(\mathbb{R})$ with $S(-x) = -S(x)$. For $S \equiv 0$, (1) reduces to the well-known particle model for the one-dimensional aggregation equation

$$\partial_t \mu_t = \partial_x(\mu_t V' * \mu_t), \quad \mu_0 = \bar{\mu} \tag{2}$$

for an initial probability measure $\bar{\mu}$ on $\mathbb{R}$. For this equation one can show that the corresponding particle approximation converges in the Wasserstein metric using the techniques of [4]. On the other hand, with time-varying weights, we expect the limit equation to include a source term, that is

$$\partial_t \mu_t - \partial_x(\mu_t V' * \mu_t) = \mu_t(S * \mu_t). \tag{3}$$

In [1], the derivative V' of the potential is assumed to be Lipschitz-continuous. We will instead consider the less regular potential $V(x) = |x|$, corresponding to the attractive 1D Coulomb potential. A helpful feature of this potential is that one may rewrite (2) as the Burgers equation in the variable F, related to the cumulative distribution of μ, for which the particle approximation corresponds exactly to Dafermos', or the front tracking, method [3] for scalar conservation laws. The particle approximation was used in [2] to show the equivalence of gradient flow solutions of (2) and entropy solutions for the corresponding Burgers equation.

Following the same idea, (3) can be rewritten as the Burgers-type balance law

$$\partial_t F - \partial_x F^2 = \mathrm{S}[F], \tag{4}$$

for an appropriate source term $\mathrm{S}[F]$. We show that the mean field limit of solutions to the particle system (1) is a solution of (3) using the framework of entropy solutions for the balance law (4). The correspondence between solutions of (1) and the front tracking algorithm suggests that this problem also lends itself well to numerical approximations.

References:

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