Title: Energy stable and linearly well-balanced numerical schemes for the non-linear Shallow Water Equations with Coriolis Force

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Abstract: The question of the accuracy of numerical schemes for hyperbolic systems with source terms around stationary solutions and/or in asymptotic regimes has been a subject of great interest over the last two decades, see the seminal works [2,11,12] in late nineties and the reference books [4,10] ten years later. In the context of geophysical flows and for colocated finite-volume methods applied to shallow water equations, a lot of works have been devoted to the accuracy around the so-called lake-at-rest equilibrium and more recently extended to nonzero velocity one dimensional stationary states, see [3] and references therein. But for large scale atmospheric or oceanographic flows, the relevant stationary state is the geostrophic equilibrium, see [21] for a general introduction to geophysical rotating fluid dynamics. The accuracy of colocated finite-volume numerical schemes around such an equilibrium was less investigated. To our knowledge, the first work in this field is due to Bouchut, Le Sommer and Zeitlin [5], see also [6, 7], but was fully accurate only for one-dimensional flows, as exhibited in [1]. Recently two independent works [15, 20] proposed IMplicit-EXplicit type schemes for fully nonlinear equations which are proven to be accurate near the geostrophic equilibrium but, due to their implicit part, need to solve a global Laplace equation at each time step. Note that there exists also a lot of works devoted to the approximation of the Coriolis term in staggered finite-difference schemes, see for example [19] for a linear analysis and [18] for the fully nonlinear case and in the finite-element framework [16].

In this work, we aim at designing explicit colocated finite-volume schemes that are proven to be accurate around the geostrophic equilibrium and stable in the nonlinear framework. Our work is based on the ideas developed in [1] where accurate and stable Godunov-type schemes were designed for the linear two-dimensional rotating wave equation but we will see here that further developments were needed to handle the nonlinear case in a conservative way. All the numerical schemes we consider belong to the AUSM family where the flux is divided in an advective part and a pressure part, see the seminal works [13, 14] and the recent review [9]. More precisely, we first introduce the system of equations under study and we characterise the geostrophic equilibrium (see below). Then, we define some discrete operators and we prove some of their properties. Equipped with these definitions, we can define some numerical schemes and study the two properties we are interested in: the decrease of the semi-discrete energy and the preservation of the geostrophic equilibrium in the linearised version. Note that in our framework the term semi-discrete will refer to quantities that are discrete in space but continuous in time. Finally, we illustrate the behaviour of the schemes for some standard test cases and we exhibit a great improvement when compared to a classic HLLC finite-volume scheme. In the following, we give a brief overview of the proposed method.

Let Ω be an open bounded domain of $\mathbb{R}^2$ and let $T > 0$. The nonlinear Shallow Water equations with Coriolis force formulated on $\Omega \times (0, T)$ read:

$$\begin{cases} \partial_t h + \operatorname{div}(h\boldsymbol{u}) = 0 \,, \\ \partial_t (h\boldsymbol{u}) + \operatorname{div}(\boldsymbol{u} \otimes h\boldsymbol{u}) + h\nabla\phi = -\omega h \boldsymbol{u}^\perp \,, \end{cases} \tag{1}$$

where h is the water height and $\boldsymbol{u} = (u_x, u_y)$ the horizontal velocity, $\boldsymbol{u}^\perp = (-u_y, u_x)$ denoting its orthogonal vector in the (x, y) plane. The Coriolis force is accounted for in the momentum equations through the angular speed ω. Following [8, 17], the pressure forces appear under a non conservative form through the scalar potential $\phi = gh$, where g is the standard gravity constant. For the sake of simplicity, a flat topography is considered in the present work[5].

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[5] In the case of a non-flat topography b, the present approach naturally extends by taking $\phi = g(h + b)$ and $E^p = \dfrac{1}{2} g h^2 + ghb$.

It is well-known that the total energy associated to System (1) decomposes as $E = E^p + E^k$ where

$$E^p = \frac{1}{2} g h^2 \quad \text{and} \quad E^k = \frac{1}{2} h \|\boldsymbol{u}\|^2$$

stand respectively for potential and kinetic energies. We recall that the energy E plays the role of a mathematical entropy associated to the hyperbolic system (1) and regular solutions satisfy the following conservation law

$$\partial_t E + \operatorname{div} \left[\left(\phi + \frac{1}{2} \|\boldsymbol{u}\|^2 \right) h \boldsymbol{u} \right] = 0 \,, \tag{2}$$

whereas for discontinuous solutions, the total energy is only non-increasing in time, see [4] Chapters 1.4 and 3.2. Equation (2) is a well-known criteria in fluid mechanics to select a unique solution to the system (1), that can have many weak solutions otherwise.

When developing numerical methods, main objectives are accuracy and stability. To get stability, a crucial objective is to build numerical approximations satisfying a discrete counterpart of (2) that ensures that the discrete energy is non-increasing. To achieve this, a general strategy is to consider a sufficient amount of numerical diffusion in the scheme. But in some physical contexts such as low Froude number regimes or near specific stationary states, these diffusive terms may considerably degrade the accuracy of the approximations and specific schemes are needed.

Here we are interested in flows around the geostrophic balance:

$$\nabla \phi + \omega \boldsymbol{u}^\perp = 0 \,, \quad \operatorname{div} \boldsymbol{u} = 0 \,. \tag{3}$$

To address such an issue, based on the study for the linear case [1], we propose a numerical approach involving discrete versions of these equilibria in the numerical fluxes. As a preliminary step, the strategy can be understood at the continuous level by investigating how Model (1) behaves with respect to some generic perturbations $(\boldsymbol{q}, \pi)$:

$$\begin{cases} \partial_t h + \operatorname{div}(h\boldsymbol{u} - \boldsymbol{q}) = 0 \,, \\ \partial_t (h\boldsymbol{u}) + \operatorname{div} \big(\boldsymbol{u} \otimes (h\boldsymbol{u} - \boldsymbol{q}) \big) + (h \nabla \phi - \nabla \pi) = -\omega \, (h\boldsymbol{u} - \boldsymbol{q})^\perp \,, \end{cases} \tag{4}$$

where $\boldsymbol{q}$ and π should be seen as continuous counterpart of numerical corrections (with respect to the flow rate and to the hydrostatic pressure) that are introduce in the numerical scheme to ensure stability property. Smooth solutions to the modified equations (4) satisfy the following energy balance:

$$\partial_t E + \operatorname{div} \left[\left(\phi + \frac{1}{2} \|\boldsymbol{u}\|^2 \right) \, (h\boldsymbol{u} - \boldsymbol{q}) - \pi \boldsymbol{u} \right] = -\boldsymbol{q} \cdot \left(\nabla \phi + \omega \boldsymbol{u}^\perp \right) - \pi \, \operatorname{div} \boldsymbol{u} \,, \tag{5}$$

which motivates a choice for $\boldsymbol{q}$ and π involving respectively the quantities $\nabla \phi + \omega \, \boldsymbol{u}^\perp$ and $\operatorname{div} \boldsymbol{u}$. Let us remark that these quantities govern the geostrophic equilibrium (3) associated to System (1) linearised around the steady state $(\tilde{h}, \widetilde{\boldsymbol{u}}) = (h_0, 0)$ for a constant h_0:

$$\begin{cases} \partial_t h = -h_0 \, \operatorname{div} \boldsymbol{u} \,, \\ \partial_t \boldsymbol{u} = -(\nabla \phi + \omega \, \boldsymbol{u}^\perp) \,. \end{cases} \tag{6}$$

From a numerical point of view, diffusion terms are thus expected to have regularising effects in the sense that they allow to recover a discrete counterpart of (5). Moreover, such terms are intended to vanish close to the geostrophic equilibrium, which must improve the quality of the approximations in this regime. We implement this idea in a discrete setting and obtain a non-conservative scheme (in the sense that the pressure forces appear under a non conservative form but the mass is nevertheless conserved) and a conservative one. Both schemes exhibit a great improvement when compared to a classic HLLC finite-volume scheme.

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