Quadratic of stability of flux limiters for transport

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Abstract

We propose a novel approach to study the quadratic stability of 2D flux limiters for non expansive transport equations.

The key result can be formulated in dimension one for the advection equation $\partial_t u + \partial_x u = 0$. A generic and well known scheme writes as

$$\frac{\overline{U}_j - U_j}{\Delta t} + \frac{U_{j+\frac{1}{2}} - U_{j-\frac{1}{2}}}{\Delta x} = 0, \qquad j \in \mathbb{Z}, \tag{1}$$

where the finite volume flux is

$$U_{j+\frac{1}{2}} = U_j + \frac{1-\nu}{2}\varphi_{j+\frac{1}{2}}(U_{j+1} - U_j) \text{ for all } j \in \mathbb{Z}. \tag{2}$$

In these notations, U_j stand for U_j^n which is the numerical solution at time step $t_n = n\Delta t$ and $\overline{U}_j$ stands for U_j^{n+1} which is the numerical solution at time step $t_{n+1} = (n+1)\Delta t$. The CFL number is $\nu = \frac{\Delta t}{\Delta x} \in (0, 1]$.

Theorem 1. *Assume the CFL condition and assume $0 \le \varphi_{j+\frac{1}{2}} \le 1$ for all $j \in \mathbb{Z}$. Then the scheme is stable in quadratic norm: $\sum_{j\in\mathbb{Z}} |\overline{U_j}|^2 \le \sum_{j\in\mathbb{Z}} |U_j|^2$. Then, under standard regularity assumptions on the exact solution, the L^2 norm at finite time of the numerical error is not less than $O(\Delta x^{\frac{1}{2}})$.*

This result admits a 2D generalization. Consider the advection equation in dimension two $\partial_t u + p\partial_x u + q\partial_y u = 0$ with $p, q \ge 0$ and $p + q = 1$. Consider the scheme

$$\frac{\overline{U}_{i,j} - U_{i,j}}{\Delta t} + p\frac{U^{\lim}_{i+\frac{1}{2},j} - U^{\lim}_{i-\frac{1}{2},j}}{\Delta x} + q\frac{U^{\lim}_{i,j+\frac{1}{2}} - U^{\lim}_{i,j-\frac{1}{2}}}{\Delta x} + \frac{pq}{2}\frac{C^{\lim}_{i-\frac{1}{2},j+\frac{1}{2}} - C^{\lim}_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta x} = 0 \tag{3}$$

where

$$\left\{ \begin{array}{l} U^{\lim}_{i+\frac{1}{2},j} = U_{i,j} + \frac{1-\nu}{2}\varphi_{i+1,j}\Delta_{i+1,j}, \\ U^{\lim}_{i,j+\frac{1}{2}} = U_{i,j} + \frac{1-\nu}{2}\varphi_{i,j+1}\Delta_{i,j+1}, \\ C^{\lim}_{i+\frac{1}{2},j+\frac{1}{2}} = \varphi_{i+\frac{1}{2},j+\frac{1}{2}}\Delta_{i+\frac{1}{2},j+\frac{1}{2}}. \end{array} \right. \tag{4}$$

Taking all limiters equal to 1, that is $\varphi_{i,j} \equiv \varphi_{i+\frac{1}{2},j+\frac{1}{2}} \equiv 1$, yields a two-dimension version of the Lax-Wendroff numerical scheme. The cell-based limiter

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$\varphi_{i,j}$ is the equivalent of $\varphi_{j+\frac{1}{2}}$ in dimension one. The corner-based limiter limiter $\varphi_{i-\frac{1}{2},j+\frac{1}{2}}$ is a new term related to the tensorial nature of numerical diffusion of the Lax-Wendroff numerical scheme in dimension two. Inspired by the quadratic properties in dimension one, we study the properties of the scheme when the flux limiters satisfy the bounds

$$0 \leq \varphi_{i,j}, \varphi_{i+\frac{1}{2},j+\frac{1}{2}} \leq 1 \qquad \text{for all } i, j. \tag{5}$$

Theorem 2. *The non linear scheme (3-5) in dimension two is quadratically stable under CFL and is convergent in quadratic norm with an order not less than* $O(\Delta x^{\frac{1}{2}})$.

These results are L^2 based and not L^1 or TVD based. That is why they offer the possibility to bypass the celebrated Goodman-Leveque obstruction theorem. Various theoretical options will be discussed and 2D numerical results will be presented to sustain the discussion.

References

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