Title: Green's function pointwise estimates for spectrally stable discrete shock profiles

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Abstract:

We consider a mono dimensional system of conservation laws

$$\begin{gathered}\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{U},\end{gathered} \tag{1}$$

where $d \in \mathbb{N}^*$, the space of states $\mathbb{U}$ is an open set of $\mathbb{R}^d$, and the flux $f : \mathbb{U} \to \mathbb{R}^d$ is a smooth function. We introduce $\mathcal{N} : \mathbb{U}^{\mathbb{Z}} \to \mathbb{U}^{\mathbb{Z}}$ a conservative one-step explicit finite difference scheme and look at the system

$$\begin{gathered}\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N} u^n, \\ u^0 \in \mathbb{U}^{\mathbb{Z}}.\end{gathered} \tag{2}$$

Discrete shock profiles are traveling waves which are solutions of (2) and link two states $u_-, u_+ \in \mathbb{U}$. The existence and stability of such solutions for all "admissible" choices of u_- and u_+ can be seen as an improved consistency property of the scheme , mainly concerning discontinuous solutions of (1). We will focus on the stability of such solutions.

The goal of my recent work has been to study the linearization $\mathscr{L}$ of the numerical scheme $\mathcal{N}$ along a steady discrete shock profile. To be more precise, we define the temporal Green's function associated to $\mathscr{L}$ for $j_0 \in \mathbb{Z}$ as

$$\begin{gathered}\mathscr{G}(0, j_0, \cdot) := \delta_l \\ \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathscr{G}(n+1, j_0, j) := \mathscr{L}\mathscr{G}(n, j_0, j).\end{gathered}$$

where $\delta_{j_0} := (\delta_{j_0,j} Id)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathcal{M}_d(\mathbb{C}))$ is the Dirac mass on j_0. In [1, Theorem 1.1], Lafitte-Godillon proved estimates on the temporal Green's function in the case where the scheme $\mathcal{N}$ is the modified Lax-Friedrichs scheme. The objective is to extend and improve these estimates for a larger class of schemes and discrete shock profiles which satisfy some suitable spectral stability condition inspired by [2].

The study is done using spatial dynamics. Via the inverse Laplace transform, we express the temporal Green's function using the spatial Green's function, which is the solution of the resolvent equation

$$\forall z \in \mathbb{C} \backslash \sigma(\mathscr{L}), \forall j_0 \in \mathbb{Z}, \quad G(z, j_0, \cdot) = (zId - \mathscr{L})^{-1} \delta_{j_0}.$$

The proof relies on a detailed analysis of the spatial Green's function with a sharp meromorphic extension near 1 through the essential spectrum of $\mathscr{L}$. It is followed by a suitable choice of contour to express the temporal Green's function with the spatial Green's function and obtain suitable bounds (similarly as in [3]).

References

[1] Pauline Godillon, *Green's function pointwise estimates for the modified Lax-Friedrichs scheme*, M2AN,Math. Model. Numer. Anal., 37(1):1–39 (2003).

[2] V. Thomée, *Stability of Difference Schemes in the Maximum-Norm*, Journal of Differential Equations 1, 273-292, (1965)

[3] K. Zumbrum, P. Howard, *Pointwise Semigroup Methods and Stability of Viscous Schock Waves*, Indiaa Univ. Math. J., 47(3):741-871, (1998)

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